

# NONCOMMUTATIVE RIESZ THEOREM AND WEAK BURNSIDE TYPE THEOREM ON TWISTED CONJUGACY

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**ABSTRACT.** The present paper consists of two parts. In the first part, we prove a noncommutative analogue of the Riesz(-Markov-Kakutani) theorem on representation of functionals on an algebra of continuous functions by regular measures on the underlying space.

In the second part, using this result, we prove a weak version of Burnside type theorem for twisted conjugacy for arbitrary discrete groups.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

**Definition 1.1.** Let  $G$  be a countable discrete group, and let  $\phi : G \rightarrow G$  be an endomorphism. Two elements  $x, x' \in G$  are said to be  $\phi$ -conjugate (or *twisted conjugate*) iff there exists a  $g \in G$  such that

$$x' = gx\phi(g^{-1}).$$

We shall write  $\{x\}_\phi$  for the  $\phi$ -conjugacy or *twisted conjugacy* class of an element  $x \in G$ . The number of  $\phi$ -conjugacy classes is called the *Reidemeister number* of  $\phi$  and is denoted by  $R(\phi)$ . If  $\phi$  is the identity map, then the  $\phi$ -conjugacy classes are the usual conjugacy classes in  $G$ .

If  $G$  is a finite group, then the classical Burnside theorem (e.g., see [11, p. 140]) says that the number of conjugacy classes of elements of  $G$  is equal to the number of equivalence classes of irreducible representations, i.e., points of the *unitary dual*  $\widehat{G}$ .

Consider an automorphism  $\phi$  of a finite group  $G$ . Then  $R(\phi)$  is equal to the dimension of the space of twisted invariant functions on  $G$ . Hence, by Peter-Weyl theorem (which asserts the existence of a two-side equivariant isomorphism  $C^*(G) \cong \bigoplus_{\rho \in \widehat{G}} \text{End}(H_\rho)$ ),  $R(\phi)$  coincides with the sum of dimensions  $d_\rho$  of the spaces of twisted invariant elements in  $\text{End}(H_\rho)$ , where  $\rho$  runs over  $\widehat{G}$  and the space of a representation  $\rho$  is denoted by  $H_\rho$ . By the Schur lemma,  $d_\rho = 1$ , if  $\rho$  is a fixed point of  $\widehat{\phi} : \widehat{G} \rightarrow \widehat{G}$ , where  $\widehat{\phi}(\rho) := \rho \circ \phi$ , and is zero otherwise. Hence,  $R(\phi)$  coincides with the number of fixed points of  $\widehat{\phi}$  (see, e.g., [5]). The purpose of the present paper is to generalize this statement to the case of infinite discrete groups (after an appropriate adaptation).

**Remark 1.2.** If  $\phi : G \rightarrow G$  is an epimorphism, then it induces the map  $\widehat{\phi} : \widehat{G} \rightarrow \widehat{G}$ ,  $\widehat{\phi}(\rho) = \rho \circ \phi$  (since a representation is irreducible if and only if the scalar operators in the representation space are the only ones which commute with all representation operators).

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This is not the case for a general endomorphism  $\phi$ , since  $\rho\phi$  may be reducible for an irreducible representation  $\rho$ , and  $\widehat{\phi}$  can be defined only as a multi-valued map. Nevertheless, we can define the set  $\text{Fix}(\widehat{\phi})$  of fixed points of  $\widehat{\phi}$  on  $\widehat{G}$  for a general endomorphism  $\phi$ .

**Definition 1.3.** Let  $\text{Rep}(G)$  be the space of equivalence classes of finite dimensional unitary representations of  $G$ . Then the corresponding map  $\widehat{\phi}_R : \text{Rep}(G) \rightarrow \text{Rep}(G)$  is defined in the same way as above:  $\widehat{\phi}_R(\rho) = \rho \circ \phi$ .

Let  $\text{Fix}(\widehat{\phi})$  be the set of points  $\rho \in \widehat{G} \subset \text{Rep}(G)$  such that  $\widehat{\phi}_R(\rho) = \rho$ .

**Theorem 1.4** (twisted Burnside theorem for type I groups [6]). *Suppose that  $G$  is a finitely generated discrete group of type I,  $\phi$  is an endomorphism of  $G$ ,  $R(\phi)$  is the number of  $\phi$ -conjugacy classes, and  $S(\phi) = \#\text{Fix}(\widehat{\phi})$  is the number of  $\widehat{\phi}$ -invariant equivalence classes of irreducible unitary representations. If  $R(\phi)$  or  $S(\phi)$  is finite, then  $R(\phi) = S(\phi)$ .*

Let  $\mu(d)$ ,  $d \in \mathbb{N}$ , be the Möbius function; i.e.,

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1, \\ (-1)^k & \text{if } d \text{ is a product of } k \text{ distinct primes,} \\ 0 & \text{if } d \text{ is not square-free.} \end{cases}$$

**Theorem 1.5** (congruences for the Reidemeister numbers [6]). *Let  $\phi : G \rightarrow G$  be an endomorphism of a countable discrete group  $G$  such that all numbers  $R(\phi^n)$  are finite, and let  $H \subset G$  be a subgroup such that  $\phi(H) \subset H$ , and for each  $x \in G$  there exists an  $n \in \mathbb{N}$  with  $\phi^n(x) \in H$ . If the pair  $(H, \phi^n)$  satisfies the assumptions of Theorem 1.4 for each  $n \in \mathbb{N}$ , then*

$$\sum_{d|n} \mu(d) \cdot R(\phi^{n/d}) \equiv 0 \pmod{n}$$

for all  $n$ .

The situation is much more complicated for groups of type  $\text{II}_1$ . For example, for the semi-direct product defined by the action of  $\mathbb{Z}$  on  $\mathbb{Z} \oplus \mathbb{Z}$  by a hyperbolic automorphism, we have found an automorphism with finite Reidemeister number (equal to four) such that  $\widehat{\phi}$  has at least five fixed points on  $\widehat{G}$  [7]. This phenomenon is due to bad separation properties of  $\widehat{G}$  for general discrete groups. A deeper study leads to the following general theorem, which is one of two main results of the present paper.

**Theorem 1.6** (weak Burnside type theorem for twisted classes). *The dimension  $R_*(\phi)$  of the space of twisted invariant functions on  $G$  lying in the Fourier-Stieltjes algebra  $B(G)$  is equal to the number  $S_*(\phi)$  of generalized fixed points  $I$  of the homeomorphism  $\widehat{\phi}$  (the sum of codimensions of the subspaces generated by elements of the form  $a - \delta_g * a * \delta_{\phi(g^{-1})} + I$ ) on the Glimm spectrum of  $G$ , i.e., on the complete regularization of  $\widehat{G}$ , provided that at least one of the numbers  $R_*(\phi)$  and  $S_*(\phi)$  is finite. Here  $\delta_g$  is the delta function supported at  $g$ .*

This result allows one to obtain the strong form  $R(\phi) = S(\phi)$  of the twisted Burnside theorem in a number of cases. The proof of the generalized Burnside theorem in [6] for groups of type I (see Theorem 1.4 in the present paper and also [5, 7]) used an identification of  $R(\phi)$  with the dimension of the space of twisted invariant ( $L^\infty$ -)functions on

$G$ , i.e., twisted invariant functionals on  $L^1(G)$ . Since only part of  $L^\infty$ -functions (namely, Fourier-Stieltjes functions) define functionals on  $C^*(G)$ , one a priori has  $R_*(\phi) \leq R(\phi)$ . Nevertheless, functions satisfying some symmetry conditions very often lie in the Fourier-Stieltjes algebra, so one can conjecture that  $R(\phi) = R_*(\phi)$  provided that  $R(\phi) < \infty$ . This is the case for all known examples.

The weak theorem will be proved as follows. The well-known Riesz(-Markov-Kakutani) theorem identifies the space of linear functionals on algebra  $A = C(X)$  with the space of regular measures on  $X$ . To prove the weak twisted Burnside theorem, we first obtain a generalization of the Riesz theorem to the case of a noncommutative  $C^*$ -algebra  $A$  using the Dauns-Hofmann theorem on representation of  $C^*$ -algebras by sections. The corresponding measures on the Glimm spectrum are functional-valued. In the extreme situations (a commutative algebra or an algebra with one-point Glimm spectrum), this theorem (which is the second main result of the present paper) either is reduced to the Riesz theorem or becomes tautological, but for the group  $C^*$ -algebras of discrete groups one in many cases obtains a new tool for counting twisted conjugacy classes.

The interest in twisted conjugacy relations has its origins, in particular, in the Nielsen-Reidemeister fixed point theory (e.g., see [10, 5]), Selberg theory (e.g., see [14, 1]), and algebraic geometry (e.g., see [9]). Note that the Reidemeister number of an endomorphism of a finitely generated Abelian group is known to be finite if and only if 1 is not in the spectrum of the restriction of this endomorphism to the free part of the group (e.g., see [10]). The Reidemeister number of any automorphism of a nonelementary Gromov hyperbolic group is infinite [8].

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## 2. ALGEBRAS OF OPERATOR FIELDS

First, we recall some facts from the theory of operator fields following [4] (see also [3, §10]). Let  $T$  be a topological space, and let a  $C^*$ -algebra (or, more generally, involutive Banach algebra)  $A_t$  be assigned to each point  $t \in T$ .

**Definition 2.1.** A *continuity structure* for  $T$  and the family  $\{A_t\}$  is a linear space  $F$  of operator fields on  $T$  ranging in  $\{A_t\}$ , (i.e., maps sending each  $t \in T$  to an element of  $A_t$ ) and possessing the following properties:

- (1) if  $x \in F$ , then the real-valued function  $t \mapsto \|x(t)\|$  is continuous on  $T$ ;
- (2) the set  $\{x(t) \mid x \in F\}$  is dense in  $A_t$  for each  $t \in T$ ;
- (3)  $F$  is closed under pointwise multiplication and involution.

**Definition 2.2.** An operator field  $x$  is said to be *continuous* with respect to  $F$  at a point  $t_0$  if for each  $\varepsilon > 0$  there exists an element  $y \in F$  and a neighborhood  $U$  of  $t_0$  such that

$\|x(t) - y(t)\| < \varepsilon$  for all  $t \in U$ . The field  $x$  is *continuous on  $T$*  if it is continuous at all points of  $T$ .

**Definition 2.3.** A *full algebra of operator fields* is a family  $A$  of operator fields on  $T$  satisfying the following conditions

- (1)  $A$  is a  $*$ -algebra, i.e., is closed under all the pointwise algebraic operations;
- (2) for each  $x \in A$ , the function  $t \mapsto \|x(t)\|$  is continuous on  $T$  and vanishes at infinity;
- (3) for each  $t$ ,  $\{x(t) \mid x \in A\}$  is dense in  $A_t$ ;
- (4)  $A$  is complete in the norm  $\|x\| = \sup_t \|x(t)\|$ .

A full algebra of operator fields is obviously a continuity structure. If  $F$  is any continuity structure, let us define  $C_0(F)$  to be the family of all operator fields  $x$  continuous on  $T$  with respect to  $F$  such that  $t \mapsto \|x\|$  vanishes at infinity. One can prove that  $C_0(F)$  is a full algebra of operator fields.

**Lemma 2.4.** *For any full algebra  $A$  of operator fields on  $T$ , the following three conditions are equivalent:*

- (1)  $A$  is a maximal full algebra of operator fields;
- (2)  $A = C_0(F)$  for some continuity structure  $F$ ;
- (3)  $A = C_0(A)$ .

Such a maximal full algebra  $A$  of operator fields may sometimes be called a *continuous direct sum* of the family  $\{A_t\}$ . We shall study the unital case; thus  $T$  is compact, and the property of vanishing at infinity is void. Moreover, we suppose that  $T$  is Hausdorff and hence normal. Clearly, in this case the full algebra is *separating* in the sense that if  $s, t \in T$ ,  $s \neq t$ ,  $\alpha \in A_s$ , and  $\beta \in A_t$ , then there exists an  $x \in A$  such that  $x(s) = \alpha$  and  $x(t) = \beta$ . We will need to distinguish an algebra  $A$  itself from its realization as the algebra  $\Gamma(\mathcal{A})$  of section of the field  $\mathcal{A} = \{A_t\}$  of algebras. We denote the section corresponding to an element  $a \in A$  by  $\hat{a}$ .

### 3. FUNCTIONALS AND MEASURES

**Definition 3.1.** Let  $\Sigma$  be an algebra of subsets of  $T$ . A *measure associated with a maximal full algebra  $A = \Gamma(\mathcal{A})$  of operator fields* is a set function  $\mu : S \mapsto \mu(S) \in \Gamma(\mathcal{A})^* = A^*$ ,  $S \in \Sigma$ , such that  $\mu(S)(a) = 0$  whenever  $\text{supp } \hat{a} \cap S = \emptyset$ . In what follows, we use the abbreviation AOFM for such a measure.

An AOFM  $\mu$  is *additive* if  $\mu(\sqcup S_i)(a) = \sum_i \mu(S_i)(a)$

An AOFM  $\mu$  is *bounded* if the supremum  $\|\mu\|$  of  $\sum_i \|\mu(S_i)\|$  over partitions  $\{S_i\}$  of  $T$  is finite.

A bounded additive AOFM will be abbreviated as BA AOFM.

**Definition 3.2.** A BA AOFM is  *$*$ -weakly regular* (a RBA AOFM) if for any  $E \in \Sigma$ ,  $a \in A$ , and  $\varepsilon > 0$  there exists a set  $F \in \Sigma$  whose closure is contained in  $E$  and a set  $G \in \Sigma$  whose interior contains  $E$  such that  $|\mu(C)a| < \varepsilon$  for every  $C \in \Sigma$  with  $C \subset G \setminus F$ .

In what follows, for  $\Sigma$  we take the algebra of all subsets of  $T$  or the algebra generated by closed subsets of  $T$ .

**Definition 3.3.** Suppose that an AOFM  $\lambda$  is defined on an algebra  $\Sigma$  of subsets of  $T$  and  $\lambda(\emptyset) = 0$ . A set  $E \in \Sigma$  is called  $\lambda$ -set if

$$\lambda(M) = \lambda(M \cap E) + \lambda(M \cap (T \setminus E))$$

for each  $M \in \Sigma$ .

**Lemma 3.4.** Suppose that  $\lambda$  is an AOFM defined on an algebra  $\Sigma$  of subsets of  $T$  and  $\lambda(\emptyset) = 0$ . The family of  $\lambda$ -sets is a subalgebra of  $\Sigma$  on which  $\lambda$  is additive. Furthermore, if  $E$  is the union of a finite set  $\{E_n\}$  of disjoint  $\lambda$ -sets and  $M \in \Sigma$ , then  $\lambda(M \cap E) = \sum_n \lambda(M \cap E_n)$ .

*Proof.* It is clear that the void set, the whole space, and the complement of any  $\lambda$ -set are  $\lambda$ -sets. Now let  $X$  and  $Y$  be  $\lambda$ -sets, and let  $M \in \Sigma$ . Since  $X$  is a  $\lambda$ -set, we have

$$(1) \quad \lambda(M \cap Y) = \lambda(M \cap Y \cap X) + \lambda(M \cap Y \cap (T \setminus X)),$$

and since  $Y$  is a  $\lambda$ -set, we have

$$(2) \quad \begin{aligned} \lambda(M) &= \lambda(M \cap Y) + \lambda(M \cap (T \setminus Y)), \\ \lambda(M \cap (T \setminus (X \cap Y))) &= \lambda(M \cap (T \setminus (X \cap Y)) \cap Y) + \lambda(M \cap (T \setminus (X \cap Y)) \cap (T \setminus Y)); \end{aligned}$$

hence

$$(3) \quad \lambda(M \cap (T \setminus (X \cap Y))) = \lambda(M \cap (T \setminus X) \cap Y) + \lambda(M \cap (T \setminus Y)).$$

It follows from (1) and (2) that

$$\lambda(M) = \lambda(M \cap Y \cap X) + \lambda(M \cap Y \cap (T \setminus X)) + \lambda(M \cap (T \setminus Y)),$$

and (3) implies that

$$\lambda(M) = \lambda(M \cap Y \cap X) + \lambda(M \cap (T \setminus (X \cap Y))).$$

Thus  $X \cap Y$  is a  $\lambda$ -set. Since  $\cup X_n = T \setminus \cap (T \setminus X_n)$ , we conclude that the  $\lambda$ -sets form an algebra. Now if  $E_1$  and  $E_2$  are disjoint  $\lambda$ -sets, then replacing  $M$  by  $M \cap (E_1 \cup E_2)$  in Definition 3.3, we see that

$$\lambda(M \cap (E_1 \cup E_2)) = \lambda(M \cap E_1) + \lambda(M \cap E_2).$$

The second assertion of the lemma follows from this by induction.  $\square$

It is well known that each functional  $\tau$  on a  $C^*$ -algebra  $B$  can be represented as a linear combination of four positive functionals in the following canonical way. First, represent  $\tau$  as  $\tau = \tau_1 + i\tau_2$ , where the self-adjoint functionals  $\tau_1$  and  $\tau_2$  are given by

$$(4) \quad \tau_1(a) = \frac{\tau(a) + \overline{\tau(a^*)}}{2}, \quad \tau_2(a) = \frac{\tau(a) - \overline{\tau(a^*)}}{2i}.$$

By the lemma on the Jordan decomposition, each self-adjoint functional  $\alpha$  can be uniquely represented as the difference  $\alpha = \alpha_+ - \alpha_-$  of two positive functionals such that

$$(5) \quad \|\alpha\| = \|\alpha_+\| + \|\alpha_-\|$$

(see [12, §3.3], [13, Theorem 3.2.5]). Let us decompose an AOFM accordingly. Since the decomposition is unique, it follows that the summands are AOFM. If we start from BA AOFM, then the additivity of the summands will follow from the uniqueness of the decomposition, and the boundedness (with a constant twice as large) will follow from (4) and property (5). The same argument shows that the summands are  $*$ -weak regular,

provided that so is the original AOFM. Thus the AOFM in the decomposition are *positive* in the sense that

$$\mu(E)(a^*a) \geq 0$$

for each  $E \in \Sigma$ . Such a set function is nondecreasing with the respect to set inclusion.

**Lemma 3.5.** *The sets  $F$  and  $G$  in the Definition 3.2 can be chosen in such a way that  $|\mu(C)(fa)| < \varepsilon$  for each continuous function  $f : T \rightarrow [0, 1]$ .*

*Proof.* Consider the decompositions  $\mu = \sum_{i=1}^4 x_i \mu_i$  and  $a = \sum_{j=1}^4 y_j a_j$ , where  $\mu_i$  and  $a_j$  are positive and  $x_i$  and  $y_j$  are complex numbers of norm  $\leq 1$ . Let us choose the sets  $F$  and  $G$  as in Definition 3.2 for  $\varepsilon/16$  and for all pairs  $\mu_i, a_j$  simultaneously. Then

$$0 \leq \mu_i(C)(f \cdot a_j) = \mu_i(C)((a_j)^{1/2} f (a_j)^{1/2}) \leq \mu_i(C)(a_j) \leq \frac{\varepsilon}{16},$$

and

$$|\mu_i(C)(f \cdot a)| \leq \sum_{j=1}^4 |x_i y_j| \cdot |\mu_i(C)(a_j)| \leq 16 \cdot \frac{\varepsilon}{16} = \varepsilon.$$

□

**Theorem 3.6.** *Let a unital separable  $C^*$ -algebra  $A$  be isomorphic to a full algebra  $\Gamma(\mathcal{A})$  of operator fields over a Hausdorff space  $X$ . Then the functionals on  $A \cong \Gamma(\mathcal{A})$  can be identified with RBA AOFM associated with  $\Gamma(\mathcal{A})$ .*

*Proof.* The assumptions of the theorem imply that  $T$  is a separable Hausdorff compact and the unit ball of the dual space of  $A$  is a metrizable compact set in the  $*$ -weak topology.

Obviously, RBA AOFM form a linear normed space with respect to  $\|\cdot\|$ .

First, we wish to prove that the natural linear map  $\mu \mapsto \mu(T)$  is an isometry of the space of RBA AOFM into  $A^*$ . Since  $\|\mu(T)\| \leq \|\mu\|$ , it is of norm  $\leq 1$ . Now take an arbitrary small  $\varepsilon > 0$ . Let  $E_1, \dots, E_n$  be a partition of  $T$  such that

$$\sum_{i=1}^n \|\mu(E_i)\| \geq \|\mu\| - \varepsilon.$$

Let  $a_i \in A$  be elements of norm 1 such that  $\mu(E_i)(a_i) \geq \|\mu(E_i)\| - \varepsilon/n$ .

By the  $*$ -weak regularity of  $\mu$  and the normality of  $T$ , there exist closed sets  $C_i$ , disjoint open sets  $G_i$ , and continuous functions  $f_i : T \rightarrow [0, 1]$  such that  $C_i \subset E_i$ ,  $|\mu(E_i \setminus C_i)(a_j)| \leq \varepsilon/n^2$ ,  $C_i \subset G_i$ ,  $|\mu(G_i \setminus C_i)(a_j)| \leq \varepsilon/n^2$  (and estimations remain valid after multiplication by positive functions as well, as in Lemma 3.5),  $f_i(s) = 0$  if  $s \notin G_i$ , and  $f_i(s) = 1$  if  $s \in C_i$ ,  $i, j = 1, \dots, n$ .

Consider the element  $a := \sum_i f_i a_i \in \Gamma(\mathcal{A}) = A$ . Then  $\|a\| \leq 1$  and

$$\begin{aligned} |\mu(S)(a) - \|\mu\|| &\leq \sum_{i=1}^n |\mu(E_i)(a) - \mu(E_i)| + \varepsilon \\ &\leq \sum_{i=1}^n |\mu(E_i \setminus C_i)(a) + \mu(C_i)(a) - \mu(E_i)(a_i)| + 2\varepsilon \\ &= \sum_{i=1}^n \left| \sum_{j=1}^n \mu(E_i \setminus C_i)(f_j a_j) + \mu(C_i)(a_i) - \mu(E_i)(a_i) \right| + 2\varepsilon \end{aligned}$$

$$\leq \sum_{i,j=1}^n |\mu(E_i \setminus C_i)(f_j a_j)| + \sum_{i=1}^n |\mu(E_i \setminus C_i)(a_i)| + 2\varepsilon \leq n^2 \frac{\varepsilon}{n^2} + n \frac{\varepsilon}{n^2} + 2\varepsilon \leq 4\varepsilon.$$

Since  $\varepsilon$  is arbitrary small, it follows that  $\|\mu\| = \|\mu(S)\|$ .

It remains to represent the general functional  $\varphi$  as an RBA AOFM. This functional on  $\Gamma(\mathcal{A})$  can be extended by the Hahn-Banach theorem to a continuous functional  $\psi$  on  $B(\mathcal{A}) = \prod_{t \in T} A_t$  (the  $C^*$ -algebra of not necessary continuous sections of  $\mathcal{A}$  with the sup-norm). This functional can be decomposed as  $\psi = \sum_{i=1}^4 \alpha_i \psi_i$ , where the  $\psi_i$  are positive functionals,  $\alpha_i \in \mathbb{C}$ ,  $|\alpha_i| = 1$ , and  $\|\psi_i\| \leq \|\psi\|$ . Let

$$\lambda(E)(a) := \psi(\chi_E a), \quad \lambda_i(a) := \psi_i(\chi_E a), \quad i = 1, \dots, 4,$$

where  $a \in \Gamma(\mathcal{A})$  and  $\chi_E$  is the characteristic function of  $E$ . Obviously,  $\lambda(T)(a) = \psi(a)$  and  $\lambda$  is a BA AOFM. Indeed, the first two properties of Definition 3.1 are obvious. The third property can be verified for each  $\lambda_i$ ,  $i = 1, \dots, 4$ :

$$\sum_{j=1}^N |\lambda_i(E_j)| = \sum_{j=1}^N \lambda_i(E_j)(\mathbf{1}) = \sum_{j=1}^N \psi_i(\chi_{E_j} \mathbf{1}) = \psi_i(\mathbf{1}) \leq \|\psi_i\|;$$

hence,

$$\sum_{j=1}^N |\lambda(E_j)| \leq \sum_{i=1}^4 \sum_{j=1}^N |\lambda_i(E_j)| \leq \sum_{i=1}^4 \|\psi_i\| \leq 4 \cdot \|\psi\|.$$

Now we wish to find an RBA AOFM  $\mu$  such that  $\mu(T)(a) = \lambda(T)(a)$ . Without loss of generality, it suffices to do this for a positive measure  $\lambda = \lambda_i$ .

Let  $F$  represent a general closed subset,  $G$  a general open subset, and  $E$  a general subset of  $T$ . Define  $\mu_1$  and  $\mu_2$  by setting

$$\mu_1(F)(a^*a) = \inf_{G \supset F} \lambda(G)(a^*a), \quad \mu_2(E)(a^*a) = \sup_{F \subset E} \mu_1(F)(a^*a),$$

and then by taking the linear extension. More precisely, owing to separability there exists a cofinal sequence  $\{G_i\}$ ,  $G_i \supset F$ . The unit ball in the dual space is weakly compact, and there exists a weakly convergent sequence  $\lambda(G_{i_k})$ . Its limit  $\psi$  is a positive functional on  $A$  enjoying the desired inf-property on positive elements. In particular, it is independent of the choice of  $\{G_i\}$  and  $\{G_{i_k}\}$ . One defines  $\mu_2$  in a similar way.

These set functions  $\mu_1$  and  $\mu_2$  are nonnegative and nondecreasing. Let  $G_1$  be open, and let  $F_1$  be closed. If  $G \supset F_1 \setminus G_1$ , then  $G_1 \cap G \supset F_1$  and  $\lambda(G_1) \leq \lambda(G_1) + \lambda(G)$ , so that  $\mu_1(F_1) \leq \lambda(G_1) + \lambda(G)$ . Since  $G$  is an arbitrary open set containing  $F_1 \setminus G_1$ , we have

$$\mu_1(F_1) \leq \lambda(G_1) + \mu_1(F_1 \setminus G_1).$$

If  $F$  is a closed set, then from this inequality, by allowing  $G_1$  to range over all open sets containing  $F \cap F_1$ , we have

$$\mu_1(F_1) \leq \mu_1(F \cap F_1) + \mu_2(F_1 \setminus F).$$

If  $E$  is an arbitrary subset of  $T$  and  $F_1$  ranges over all closed subsets of  $E$ , then it follows from the preceding inequality that

$$(6) \quad \mu_2(E) \leq \mu_2(E \cap F) + \mu_2(E \setminus F).$$

We claim that

$$(7) \quad \mu_2(E) \geq \mu_2(E \cap F) + \mu_2(E \setminus F)$$

for an arbitrary subset  $E$  of  $T$  and arbitrary closed set  $F$  in  $T$ . To see this, let  $F_1$  and  $F_2$  be disjoint closed sets. Since  $T$  is normal, it follows that there are disjoint neighborhoods  $G_1$  and  $G_2$  of  $F_1$  and  $F_2$ , respectively. If  $G$  is an arbitrary neighborhood of  $F_1 \cup F_2$ , then  $\lambda(G) \geq \lambda(G \cap G_1) + \lambda(G \cap G_2)$ , so that

$$\mu_1(F_1 \cap F_2) \geq \mu_1(F_1) + \mu_2(F_2).$$

We now let  $E$  and  $F$  be arbitrary sets in  $T$  with  $F$  closed, and let  $F_1$  range over closed subsets of  $E \cap F$  while  $F_2$  ranges over the closed subsets of  $E \setminus F$ . The preceding inequality then proves (7). From (6) and (7), we have

$$(8) \quad \mu_2(E) = \mu_2(E \cap F) + \mu_2(E \cap (T \setminus F))$$

for an arbitrary  $E$  in  $T$  and a closed subset  $F \subset T$ . The function  $\mu_2$  is defined on the algebra of all subsets of  $T$ , and it follows from (8) that each closed set  $F$  is a  $\mu_2$ -set. If  $\mu$  is the restriction of  $\mu_2$  to the algebra generated by closed sets, then it follows from Lemma 3.4 that  $\mu$  is additive on this algebra. It is clear from the definition of  $\mu_1$  and  $\mu_2$  that  $\mu_1(F) = \mu_2(F) = \mu(F)$  if  $F$  is closed, and hence  $\mu(E) = \sup_{F \subset E} \mu(F)$ . This shows that  $\mu$  is \*-weak regular, and since  $\|\mu(T)\| < \infty$ , we see that  $\mu$  is RBA AOFM.

Finally, by definition,  $\mu(S)(a) = \lambda(S)(a) = \psi(a) = \varphi(a)$  for  $a \in \Gamma(\mathcal{A})$ .  $\square$

#### 4. TWISTED-INVARIANT AOFM

The most part of the subsequent argument is valid for various representations of algebras by operator fields, but for now we restrict ourselves to the case of the group  $C^*$ -algebra of a discrete group and concentrate ourselves on the following important representation by sections due to Dauns and Hofmann [2, Corollaries 8.13, 8.14].

Let  $Z$  be the center of a  $C^*$ -algebra  $A$ , and let  $\widehat{Z}$  be the space of maximal ideals of  $Z$  equipped with the standard topology. If  $I \in \mathcal{P} := \text{Prim}(A)$  (the space of kernels of unitary irreducible representations), then  $Z \cap I \in \widehat{Z}$ . (This follows from the fact that the restriction to  $Z$  of an irreducible representation with kernel  $I$  gives rise to a homomorphism  $Z \rightarrow \mathbb{C}$  and hence  $Z \cap I$  is a maximal ideal.) We obtain a map  $f : \mathcal{P} \rightarrow \widehat{Z}$ . Suppose that  $T := f(\mathcal{P})$ . For each  $x \in T$ , consider the ideal  $I_x := \cap I$ ,  $f(I) = x$ , (the *Glimm ideal*) and the field  $A/I_x$  of algebras. We have the map  $a \mapsto \{x \mapsto a + I_x\}$  of the algebra  $A$  into the algebra of sections of this field. An important result in [2] is that this map is an isomorphism. The map  $f : \mathcal{P} \rightarrow T$  is universal with respect to continuous maps  $g : \mathcal{P} \rightarrow S$  to Hausdorff spaces, i.e., any such map can be represented as  $h \circ f$  for some continuous  $h : T \rightarrow S$ . The space  $T$  is compact for a unital algebra.

Now we consider a countable discrete group  $G$  and an automorphism  $\phi$  of  $G$ . Let  $A = C^*(G)$ . One has the twisted action of  $G$  on  $A$ ,

$$g[a] = \delta_g * a * \delta_{\phi(g^{-1})},$$

where  $\delta_g$  is the delta function supported at  $g$ , and a similar action on functionals, since they are realized as some functions on  $G$ . (It coincides with the dual action up to the replacement  $g \rightarrow g^{-1}$ .) The same action is defined on  $A/I$  since the ideals are shift invariant.



**Definition 4.1.** The dimension  $R_*(\phi)$  of the space of twisted invariant functionals on  $C^*(G)$  is called the *generalized Reidemeister number* of  $\phi$ . Hence  $R_*(\phi)$  is the dimension of the space of twisted invariant elements of the Fourier-Stieltjes algebra  $B(G)$ .

Recall that the Fourier-Stieltjes algebra of a discrete group  $G$  has the following three equivalent definitions: (1) the space of coefficients of all unitary representations of  $G$  (i.e., functions of the form  $g \mapsto \langle \rho(g)\xi, \eta \rangle$ ,  $\xi, \eta \in H_\rho$ , where  $H_\rho$  is the space of a unitary representation  $\rho$ ); (2) the space of all finite linear combinations of positively definite functions on  $G$ ; (3) the space of bounded linear functionals on  $C^*(G)$ . The commutative multiplication in these function spaces on  $G$  is introduced pointwise.

**Definition 4.2.** A (Glimm) ideal  $I$  is a *generalized fixed point* of  $\hat{\phi}$  if the linear span of elements  $b - g[b]$  is not dense in  $A_I = A/I$ , where  $g[\cdot]$  is the twisted action, i.e., its closure  $K_I$  does not coincide with  $A_I$ . Denote by  $GFP$  the set of all generalized fixed points.

If we have only finitely many such fixed points, then the twisted invariant RBA AOFM are concentrated in these points. Indeed, let us describe the action of  $G$  on RBA AOFM in more detail. The action of  $G$  on measures is defined by the identification of measures with functionals on  $A$ .

**Lemma 4.3.** *If  $\mu$  corresponds to a twisted invariant functional, then for each Borel  $E \subset T$  the functional  $\mu(E)$  is twisted invariant.*

*Proof.* This is an immediate consequence of  $*$ -weak regularity. Indeed, suppose that  $a \in A$ ,  $g \in G$ ,  $\varepsilon > 0$  is an arbitrary small number, and  $U$  and  $F$  are defined as in Lemma 3.5, for  $a$  and  $g[a]$  simultaneously. Take a continuous function  $f : T \rightarrow [0, 1]$  with  $f|_F = 1$  and  $f|_{T \setminus U} = 0$ . Then

$$\begin{aligned} |\mu(E)(a - g[a])| &= |\mu(E \setminus F)(a) + \mu(F)(a) - \mu(E \setminus F)(g[a]) - \mu(F)(g[a])| \\ &\leq |\mu(F)(a) - \mu(F)(g[a])| + 2\varepsilon = |\mu(F)(fa) - \mu(F)(g[fa])| + 2\varepsilon \\ &= |\mu(U)(fa) - \mu(U \setminus F)(fa) - \mu(U)(g[fa]) + \mu(U \setminus F)(g[fa])| + 2\varepsilon \\ &\leq |\mu(U)(fa) - \mu(U)(g[fa])| + 4\varepsilon = |\mu(T)(fa) - \mu(T)(g[fa])| + 4\varepsilon \\ &= 4\varepsilon. \end{aligned}$$

□

**Lemma 4.4.** *For each twisted-invariant functional  $\varphi$  on  $C^*(G)$ , the corresponding measure  $\mu$  is concentrated on the set  $GFP$  of generalized fixed points.*

*Proof.* Let  $\|\mu\| = 1$ . Suppose opposite: there exists an element  $a \in A$ ,  $\|a\| = 1$ , vanishing at the generalized fixed points and satisfying  $\varphi(a) \neq 0$ . Let  $\varepsilon := |\varphi(a)| > 0$ . In each point  $t \notin GFP$  we can find elements  $b_t^i \in A$ ,  $g_t^i \in G$ ,  $i = 1, \dots, k(t)$ , such that

$$\|a(t) - \sum_{i=1}^{k(t)} (g_t^i[b_t^i](t) - b_t^i(t))\| < \varepsilon/4.$$

Then there exists a neighborhood  $U_t$  of  $t$  such that

$$\|a(s) - \sum_{i=1}^{k(t)} (g_t^i[b_t^i](s) - b_t^i(s))\| < \varepsilon/2$$

for  $s \in U_t$ . Take a finite subcover  $\{U_{t_j}\}$ ,  $j = 1, \dots, n$ , of  $\{U_t\}$  and a Borel partition  $E_1, \dots, E_n$  subordinated to this subcover. Then

$$\varphi(a) = \sum_{j=1}^n \mu(E_j)(a) = \sum_{j=1}^n \mu(E_j) \left( a - \sum_{i=1}^{k(t_j)} (g_{t_j}^i[b_{t_j}^i] - b_{t_j}^i) \right) + \sum_{j=1}^n \sum_{i=1}^{k(t_j)} \mu(E_j)(g_{t_j}^i[b_{t_j}^i] - b_{t_j}^i).$$

By Lemma 4.3, each summand in the second term is zero. The absolute value of the first term does not exceed  $\sum_j \|\mu(E_j)\| \varepsilon/2 \leq \|\mu\| \cdot \varepsilon/2 = \varepsilon/2$ . This contradicts the fact that  $|\varphi(a)| = \varepsilon$ .  $\square$

Since a functional  $\varphi$  on  $A_I$  is twisted-invariant if and only if  $\text{Ker } \varphi \supset K_I$ , it follows that the dimension of the space of these functionals is equal to the dimension of the space of functionals on  $A_I/K_I$  and is finite if and only if the space  $A_I/K_I$  is finite-dimensional. In this case, the dimension of the space of twisted invariant functionals on  $A_I$  is equal to  $\dim(A_I/K_I)$ .

**Definition 4.5.** The number

$$S_*(\phi) := \sum_{I \in GFP} \dim(A_I/K_I)$$

is called the *generalized number  $S_*(\phi)$  of fixed points* of  $\widehat{\phi}$  on the Glimm spectrum.

Since functionals associated with measures concentrated at distinct points are linearly independent (the space is Hausdorff), we see that the argument above gives the following statement.

**Theorem 4.6** (weak generalized Burnside theorem).

$$R_*(\phi) = S_*(\phi)$$

*provided that one of these numbers is finite.*

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